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Relation between Bäcklund transformations and higher continuous symmetries of the Toda equation

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Abstract

In this paper we study one aspect of the continuous symmetries of the Toda equation. Namely, we establish a correspondence between Bäcklund transformations and continuous symmetries of the Toda equation. A symmetry transformation acting on a solution of the Toda equation can be seen as a superposition of Bäcklund transformations. Conversely, a Bäcklund transformation can be written, at least formally, as a composition of infinitely many higher symmetry transformations. This result reinforces the opinion that the presence of sufficiently many continuous symmetries for discrete equations is an indication of their integrability.

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1. Introduction

This paper is part of a programme, the aim of which is to define the concept of Lie, or continuous symmetry of a difference or differential-difference equation, and to further study the properties and applications of these symmetries [1–19]. It has already been pointed out [3, 8, 11, 14] that the essential nature of most continuous symmetries on a lattice is that they transform the fields in a highly non-local manner. The transformed value of a field at a point depends on the values of the field at many other points, usually extending to the entire lattice. This is so even in the case of symmetries that in the continuous limit become point symmetries. The surprising example of the lattice continuous translation symmetry [2] illustrates the general situation, and thus standard continuous symmetries of discrete equations are naturally analogous to generalized symmetries of differential equations [20].

It is generally accepted that there exists a relation between the existence of generalized symmetries of a differential equation and its integrability [21]. This relation appears in the

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discrete case too and the existence of a non-trivial continuous symmetry in a discrete system is an indication that it may be integrable. Some works leading to this conclusion and making use of it, are [16, 17, 19], where symmetry algebras of integrable difference and differential-difference systems were described and studied. In the present paper, another observation in this general direction is made. We find a relation between Bäcklund transformations, typical of integrable equations, and the existence of infinitely many continuous symmetries for the Toda equation. Surprisingly this relation, which was observed once by Sato in a continuous limit of the Toda equation, the KdV equation, happens to be much more transparent and direct in the discrete case.

In section 2 we present notation and definitions, and describe the symmetry algebra of the Toda equation, as well as its Bäcklund transformations. In section 3 we show the relationship between Bäcklund transformations and symmetries of the Toda equation. Finally, in section 4 we draw some conclusions.

2. Higher Lie symmetries of the Toda equation

We reproduce here some known results [17, 19] to make this paper self-contained. We study the Toda equation

$$\ddot{u}_n = \mathrm{e}^{u_{n-1}-u_n} - \mathrm{e}^{u_n-u_{n+1}} \tag{1}$$

that we can better write in evolution form as the Toda system

$$\dot{a}_n = a_n(b_n - b_{n+1}) \qquad \dot{b}_n = a_{n-1} - a_n. \tag{2}$$

Equations (1) and (2) are equivalent under the identification

$$b_n = \dot{u}_n \qquad a_n = \mathrm{e}^{u_n-u_{n+1}}. \tag{3}$$

A basis for the set of so-called isospectral symmetries of (2) is generated by the flows

$$\begin{pmatrix} a_{n,\epsilon_k} \\ b_{n,\epsilon_k} \end{pmatrix} = \mathcal{L}^k \begin{pmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{pmatrix} \qquad k = 0, 1, 2, \ldots \tag{4}$$

and a general isospectral symmetry can be written as

$$\begin{pmatrix} a_{n,\epsilon} \\ b_{n,\epsilon} \end{pmatrix} = \phi(\mathcal{L}) \begin{pmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{pmatrix} \tag{5}$$

where ϕ is an entire function of its argument and $\mathcal{L}$, the recursion operator, is given by

$$\mathcal{L} \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_n b_{n+1} + a_n(q_n + q_{n+1}) + (b_n - b_{n+1})s_n \\ b_n q_n + p_n + s_{n-1} - s_n \end{pmatrix}. \tag{6}$$

Here s_n is the solution of the non-homogeneous first-order equation

$$s_{n+1} = \frac{a_{n+1}}{a_n}(s_n - p_n) \qquad \lim_{|n|\to\infty} s_n = 0. \tag{7}$$

The flows (4) form a basis of the space of all flows (5), and the integer k is the order of the basic generalized symmetry (4).

The standard way to deduce the previous equations is to make use of the integrability properties of the Toda equation [17, 22]. We use the spectral transform to relate the studied nonlinear evolution equations to linear equations for the reflection coefficient. The Toda equation is associated with the discrete Schrödinger spectral problem

$$\psi(n-1, t; \lambda) + b_n\psi(n, t; \lambda) + a_n\psi(n+1, t; \lambda) = \lambda\psi(n, t; \lambda). \tag{8}$$

For any equation on the family (4) there is an explicit evolution equation for the function $\psi(n, t; \lambda)$ [23, 24] such that λ does not evolve in ϵ_k. If we impose the boundary conditions

$$\lim_{|n|\to\infty} a_n - 1 = \lim_{|n|\to\infty} b_n = 0 \tag{9}$$

on the fields a_n and b_n, we can associate to equation (8) a spectrum defined in the complex plane of the variable z ($\lambda = z^{-1} + z$):

$$\{R(z, t),\ z \in \mathbb{C}_1; z_j, c_j(t),\ |z_j| < 1, j = 1, 2, \ldots, N\} \tag{10}$$

where $R(z, t)$ is the reflection coefficient, $\mathbb{C}_1$ is the unit circle in the complex z-plane, z_j are isolated points inside the unit disc and $c_j(t)$ are some complex functions of t, related to the residues of $R(z, t)$ at the poles z_j. When a_n and b_n satisfy the boundary conditions, the spectral data define the potentials in a unique way. Thus, there is a *one-to-one correspondence* between the evolution of asymptotically well behaved potentials (a_n, b_n) of the discrete Schrödinger spectral problem (8), satisfying the Toda system (2) and that of the reflection coefficient $R(\lambda, t)$, given by

$$\frac{dR(z, t)}{dt} = \mu R(z, t) \qquad \mu = \frac{1}{z} - z \tag{11}$$

where $\frac{\mathrm{d}}{\mathrm{d}y}$ denotes the total derivative with respect to y. The evolution in the spectral space is, according to (11), linear, much simpler than in the space of the fields.

The basic symmetry flows (4) and the generic flow (5) induce an analogous symmetry evolution on the reflection coefficient in the spectral space. For the generic case (5) we have

$$\frac{\mathrm{d}R(\lambda, \epsilon)}{\mathrm{d}\epsilon} = \mu\phi(\lambda)R(\lambda, \epsilon). \tag{12}$$

In the general case of evolution (5), i.e. when $\phi(\mathcal{L})$ is not constant and thus we deal with a generalized symmetry of (2), the flow cannot be integrated. However, the flows of generalized symmetries can be trivially integrated in the spectral space, as can be seen from (12).

In addition to symmetry transformations, the Toda system admits another family of transformations, the so-called Bäcklund transformations. They are discrete transformations (i.e. mappings) that applied to a given solution of the system produce another solution. Bäcklund transformations enjoy special properties such as commutativity, allowing the definition of a nonlinear superposition principle that endows the evolution equation with an integrability feature. The spectral transform technique [22] makes it possible to write down families of Bäcklund transformations for integrable differential equations. In the discrete case, for the example of the discrete Schrödinger spectral problem the appropriate developments are found in [25]. We have that

$$\gamma(\Lambda)\begin{pmatrix} \tilde{a}(n) - a(n) \\ \tilde{b}(n) - b(n) \end{pmatrix} = \delta(\Lambda)\begin{pmatrix} \tilde{\Pi}(n)\Pi^{-1}(n{+}1)(\tilde{b}(n) - b(n{+}1)) \\ \tilde{\Pi}(n{-}1)\Pi^{-1}(n) - \tilde{\Pi}(n)\Pi^{-1}(n{+}1) \end{pmatrix} \tag{13}$$

where $\gamma(z)$ and $\delta(z)$ are entire functions of their argument and we have denoted

$$\Pi(n) = \prod_{j=n}^{\infty} a(j) \qquad \tilde{\Pi}(n) = \prod_{j=n}^{\infty} \tilde{a}(j). \tag{14}$$

$(a_{(n)}, b_{(n)})$ and $(\tilde{a}_{(n)}, \tilde{b}_{(n)})$ are two different solutions of the Toda equation (2). Λ is the recursion operator

$$\Lambda \begin{bmatrix} p(n) \\ q(n) \end{bmatrix} = \begin{bmatrix} \tilde{a}(n)S(n+z) - a(n)S(n) + \tilde{b}(n)p(n) \\ \quad +\Sigma(n)\dfrac{\tilde{\Pi}(n)}{\Pi(n+1)}[\tilde{b}(n) - b(n+1)] \\ p(n-1) + \tilde{b}(n)q(n) - \dfrac{\tilde{\Pi}(n)}{\Pi(n+1)}\Sigma(n) + \dfrac{\tilde{\Pi}(n-1)}{\Pi(n)}\Sigma(n-1) \\ \quad +[b(n) - \tilde{b}(n)]S(n) \end{bmatrix} \tag{15}$$

with

$$S(n+1) - S(n) = q(n) \qquad \Sigma(n+1) - \Sigma(n) = -\frac{\Pi(n+2)}{\tilde{\Pi}(n+1)}p(n+1). \tag{16}$$

It is shown in [25] that in the spectral space, whenever $(a_{(n)}, b_{(n)})$ and $(\tilde{a}_{(n)}, \tilde{b}_{(n)})$ satisfy the asymptotic conditions (9), the family of Bäcklund transformations (13) transforms the reflection coefficient according to the simple expression

$$\tilde{R}(\lambda) = \frac{\gamma(\lambda) - \delta(\lambda)z}{\gamma(\lambda) - \delta(\lambda)/z} R(\lambda). \tag{17}$$

The usual example encountered in the literature is the so-called one-soliton Bäcklund transformation, that is the one corresponding to the functions $\gamma(\lambda)$ and $\delta(\lambda)$ being constant (say 1 and δ_0 respectively). We can rewrite the one-soliton Bäcklund transformation directly for the Toda equation (1) using the identification (3)

$$\dot{u}(n) - \dot{u}(n{+}1) = \frac{1}{\delta_0}\left[e^{u(n+1)-\tilde{u}(n+1)} - e^{u(n)-\tilde{u}(n)}\right] + \delta_0\left[e^{\tilde{u}(n)-u(n+1)} - e^{\tilde{u}(n-1)-u(n)}\right]. \tag{18}$$

Equation (18) is written in such a way that it can be interpreted as a (nonlinear) three-point difference equation for $\tilde{u}_n$, where u_n is some chosen solution of (1). The relevant remark here is that formulae (13) and (17) represent much more general transformations, i.e. higher-order Bäcklund transformations. If the arbitrary functions $\gamma(\lambda)$ and $\delta(\lambda)$ are polynomials, then we have a finite-order Bäcklund transformation that can be interpreted as a composition of a finite number of one-soliton transformations. In more general cases we face an infinite-order Bäcklund transformation.

Our aim in the following section is to discuss how Bäcklund transformations are related to continuous symmetry transformations, allowing, albeit formally, an integration of the latter.

3. Relation between Bäcklund transformations and higher symmetries

The purpose of this section is to present two new results, connecting Bäcklund transformations and symmetries for the Toda equation. We formulate them both as theorems.

Theorem 1. *Let*

$$R(\lambda, \epsilon) = \exp(\epsilon\mu\phi(\lambda))R(\lambda, 0) \tag{19}$$

be a symmetry in spectral parameter space. This symmetry determines a Bäcklund transformation (17), with

$$\delta(\lambda) = \frac{2}{\mu}\sinh(\epsilon\mu\phi(\lambda)) \tag{20}$$

$$\gamma(\lambda) = 1 + \cosh(\epsilon\mu\phi(\lambda)) + \frac{\lambda}{\mu}\sinh(\epsilon\mu\phi(\lambda)). \tag{21}$$

Proof. The transformation (19) of the reflection coefficient $R(\lambda)$ is obtained by integrating, at least formally, equation (12) in which $\phi(\lambda)$ is an arbitrary entire function. In order to identify the finite transformation (19) with a general Bäcklund transformation (17) of the reflection coefficient, we equate

$$R(\lambda,\epsilon)=\tilde{R}(\lambda) \qquad R(\lambda,0)=R(\lambda). \tag{22}$$

Using the relations

$$\lambda=\frac{1}{z}+z \qquad \mu=\frac{1}{z}-z \tag{23}$$

which imply

$$\mu^2=\lambda^2-4 \tag{24}$$

we obtain

$$\mathrm{e}^{\epsilon\mu\phi(\lambda)}=\frac{2-(\lambda-\mu)\beta(\lambda)}{2-(\lambda+\mu)\beta(\lambda)} \qquad \beta(\lambda)=\frac{\delta(\lambda)}{\gamma(\lambda)}. \tag{25}$$

We need to prove that $\beta(\lambda)$, defined in equation (25), depends only on λ and is the ratio of two entire functions. We have

$$\mathrm{e}^{\epsilon\mu\phi(\lambda)}=E_0(\lambda)+\mu E_1(\lambda) \tag{26}$$

$$E_0=\cosh(\epsilon\mu\phi(\lambda)) \qquad E_1(\lambda)=\frac{1}{\mu}\sinh(\epsilon\mu\phi(\lambda)). \tag{27}$$

Indeed, the power expansions of $\cosh(\epsilon\mu\phi(\lambda))$ and $\sinh(\epsilon\mu\phi(\lambda))$ contain only even and odd powers, respectively. In view of equation (24), the even powers of μ are power series in λ, the odd powers can be replaced by $\mu h(\lambda)$, where $h(\lambda)$ is a power series of λ alone. Hence both $E_0(\lambda)$ and $E_1(\lambda)$ are independent of μ and are entire functions. From equation (25) we obtain two equations for $\beta(\lambda)$

$$(2-\lambda\beta(\lambda))E_0-(\lambda^2-4)\beta(\lambda)E_1=(2-\lambda\beta(\lambda)) \tag{28}$$

$$-\beta E_0+(2-\lambda\beta)E_1=\beta. \tag{29}$$

Equations (28) and (29) are compatible (see equations (27) and (24)) and give the final result, namely equations (20) and (21). □

Theorem 2. *Let*

$$\tilde{R}(\lambda)=\frac{1-\delta z}{1-\delta/z}R(\lambda) \tag{30}$$

where δ is a real constant parameter, be an elementary Bäcklund transformation in the spectral parameter space. Then a symmetry transformation of the reflection coefficient is given by equation (19) with

$$\phi(\lambda)=\frac{1}{\epsilon\mu}\sinh^{-1}\left[\mu\frac{\delta(2-\delta\lambda)}{2(\delta^2-\delta\lambda+1)}\right]. \tag{31}$$

Proof. We need to show that $\phi(\lambda)$ in equation (31) does not depend on μ and is an entire function of λ. Expression (31) itself is obtained from (28) and (29). The reflection coefficient is defined for $z = e^{ik}$ on the unit circle. Hence it suffices to show that $\phi(\lambda)$ is entire for

$$\lambda = 2\cos(k) \qquad \mu = -2i\sin(k) \qquad 0 \leqslant k < 2\pi. \tag{32}$$

Let us denote the argument in (31)

$$\sigma = -2i\delta \frac{\sin(k)(1 - \delta\cos(k))}{\delta^2 - 2\delta\cos(k) + 1}. \tag{33}$$

First of all, σ is always finite for $\delta \in \mathbb{R}$. Indeed, the denominator in (33) vanishes only for $\delta = e^{\pm ik}$. This is real only for $k = 0$ and π. However, these two zeros of the denominator cancel with zeros of the numerator and σ remains finite, i.e. σ is an entire function.

For $|\sigma|^2 < 1$ we can expand the inverse hyperbolic function in equation (31) into a power series. The expression μ cancels. All even powers of μ are eliminated using equation (24) and we find that ϕ is an entire function of λ alone. From equation (33) we see that we have

$$\sigma(\delta = 0) = 0 \qquad \lim_{\delta\to\infty} \sigma = 1 \qquad |\sigma|^2(\delta) \leqslant 1. \tag{34}$$

□

For $\epsilon \to 0$, we have that $\delta \to 0$, so that the identity transformation in the group corresponds to a trivial Bäcklund transformation (no transformation).

4. Conclusions

In this paper we have seen how, in the case of the Toda equation, a Bäcklund transformation can be written as a generalized symmetry and vice versa.

In the case of the Toda equation, finite-order Bäcklund transformations correspond to infinite-order symmetry transformations, and vice versa, finite-order symmetries correspond to infinite-order Bäcklund transformations. It does not seem possible to find an equivalent pair of finite-order transformations. This can be compared with the situation with Fourier analysis for linear equations. If we expand a function $f(x)$ in Fourier modes, then a translated function $f(x + x_0)$ can also be expanded, but all modes are modified. Here the modes are n-soliton states and each one obtains a contribution from the symmetry transformation.

Bäcklund transformations can be composed, giving rise to higher-order Bäcklund transformations. In fact, composing the one-soliton Bäcklund transformation, we can construct an infinite series of n-soliton transformations, and ultimately any infinite-order Bäcklund transformation (17) as a formal infinite composition. We have shown that the basic one-soliton Bäcklund transformation is equivalent to a given infinite-order generalized symmetry, developable as a series of basic symmetries (4). The presence of this special higher symmetry implies the existence of all Bäcklund transformations (17), and somehow implies the existence of all the other higher symmetries.

Summarizing we conclude that, for the Toda equation, the existence of a Bäcklund transformation is inextricably related to the existence of infinitely many basic generalized symmetries. Both situations are customary features or even definitions of integrability.

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